LETTER TO THE EDITOR

Fractons and Luttinger liquids

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Abstract. We consider the concept of fractons as particles or quasiparticles which obey a specific fractal statistics in connection with a one-dimensional Luttinger liquid theory. We obtain a dual statistics parameter $\tilde{\nu} = \nu + 1$ which is identified with the controlling parameter $e^{-2\varphi}$ of the Luttinger model. In this way, a bosonic system characterized by a fractal index $i_f[h] = i_f[2] = 1$ is considered in a conformal field theory with central charge $c[\nu = 0] = 1 = i_f[2]$ with a compactified radius $R = 1/\sqrt{\tilde{\nu}} = 1$. Thus, we have a mapping of a bosonic theory to a fermionic one and vice versa, i.e. the duality symmetry $\tilde{h} = 3 - h$ of the universal class h of fractons defined in the interval $1 < h < 2$ is satisfied.

We have introduced in [1] the concept of universal classes of particles or quasiparticles, labelled by a fractal parameter (or Hausdorff dimension) h, defined in the interval $1 < h < 2$. These particles, termed *fractons*†, obey *fractal statistics* (the distribution functions are *fractal functions*):

$$n[h] = \frac{1}{\mathcal{Y}[\xi] - h} \tag{1}$$

where the function $\mathcal{Y}[\xi]$ satisfies the equation

$$\xi = \{\mathcal{Y}[\xi] - 1\}^{h-1} \{\mathcal{Y}[\xi] - 2\}^{2-h} \tag{2}$$

and $\xi = \exp\{(\epsilon - \mu)/KT\}$ has the usual definition. The particles are collected in each class taking into account the *fractal spectrum*

$$
\begin{aligned}
h - 1 &= 1 - \nu & 0 < \nu < 1 \\
h - 1 &= \nu - 1 & 1 < \nu < 2
\end{aligned}
\tag{3}
$$

etc and the spin-statistics relation $\nu = 2s$.

Fractons are charge-flux systems which 'live' in two-dimensional multiply connected space and carry rational or irrational quantum numbers, such as charge and spin. The fractal parameter h is associated with the path of the quantum-mechanical particle and so we have a *quantum-geometrical* description of the statistical laws of Nature. This means that the fractal characteristic of the quantum path, which reflects the Heisenberg uncertainty principle, is embodied in the fractal statistics expression. Thus, our formulation generalizes *in a natural way* the bosonic and fermionic distributions. Besides this, we verify that the classes h of particles have a *duality symmetry* defined by $\tilde{h} = 3 - h$, such that fermions ($h = 1$) and bosons ($h = 2$) are dual objects. As a result, we extract a *fractal supersymmetry* made explicit by the

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† This word was used by Alexander and Orbach in another context [8].

following *theorem*: *If the particle with spin s is within the class h, then its dual $s + \frac{1}{2}$ is within the class $\tilde{h}$.*

Now in [2] we have obtained a relation between fractons and conformal field theory (CFT) quasiparticles (edge excitations), through the concept of a *fractal index* associated with the classes h:

$$i_f[h] = \frac{6}{\pi^2} \int_{\infty\,(T=0)}^{1\,(T=\infty)} \frac{\mathrm{d}\xi}{\xi} \ln\{\Theta[\mathcal{Y}(\xi)]\} \tag{4}$$

where

$$\Theta[\mathcal{Y}] = \frac{\mathcal{Y}[\xi] - 2}{\mathcal{Y}[\xi] - 1} \tag{5}$$

is the single-particle partition function of the universal class h and hence

$$n[h] = \xi \frac{\partial}{\partial \xi} \ln \Theta[\mathcal{Y}]. \tag{6}$$

We have verified a connection between the central charge $c[\nu]$ (a dimensionless number which characterizes conformal field theories in two dimensions) and the universal classes h of particles. For that we considered the particle–hole duality $\nu \longleftrightarrow 1/\nu$ for integer values ν of the statistical parameter which some condensed matter systems exhibit—for example, the Calogero–Sutherland model [3] and others [4]. The central charge for ν *even* is defined by

$$c[\nu] = i_f[h, \nu] - i_f\left[h, \frac{1}{\nu}\right] \tag{7}$$

and for ν *odd* it is defined by

$$c[\nu] = 2 \times i_f[h, \nu] - i_f\left[h, \frac{1}{\nu}\right] \tag{8}$$

where $i_f[h, \nu]$ means the fractal index of the universal class h which contains the particles with distinct spin values which obey a specific fractal statistics.

In this short note we show the connection between fractons and Luttinger liquids, taking into account the parameters which characterize these models. They are one-dimensional interacting many-body systems, which at low energy or low temperature are characterized by a controlling parameter, $\mathrm{e}^{-2\varphi}$ [5].

We have also obtained a Fermi velocity given by $v_F = v_N/\tilde{\nu}$, where $\tilde{\nu} = \nu + 1$ ($\nu \geqslant 0$) is a dual statistical parameter and

$$v_N = \frac{2\hbar}{mL} c[\nu]$$

is an excitation related to the change in the particle number which depends on the central charge [2]. The quantity $\hbar$ is the Planck constant, m is the mass of the particle and L is the length of the edge. Another excitation related to the current is defined as $v_J = v_F/\tilde{\nu}$. Haldane [5] just considered a relation between these two types of excitation to characterize the Luttinger liquid: $v_F = \sqrt{v_N v_J}$. Thus, our definition satisfies this condition and, in this way, we identify the controlling parameter with the dual statistical parameter $\tilde{\nu} = \mathrm{e}^{-2\varphi}$, such that $v_N = v_F \mathrm{e}^{-2\varphi}$ and $v_J = v_F \mathrm{e}^{2\varphi}$.

Now, we observe that $\tilde{\nu} = \nu + 1$ ($\nu \geqslant 0$) satisfies the duality symmetry, i.e. we can map a fermionic theory to a bosonic one and vice versa. We also know that the fractal index $i_f[h]$ coincides with the central charge $c[\nu]$, for $0 \leqslant \nu \leqslant 1$, and we have distinguished two concepts of central charge, one related to the universal classes h and the other related to the particles which belong to these classes [2]. Thus, the fractal index for $\nu = 0$ (boson, ν even)

is $i_f[h, \nu] = i_f[2, 0] = 1$ and for the dual statistical parameter $\tilde{\nu} = 1$ (fermion, $\tilde{\nu}$ odd) it is $i_f[\tilde{h}, \tilde{\nu}] = i_f[1, 1] = \frac{1}{2}$. In this way, we define for a bosonic theory $i_f[2] = 1$ identified with a conformal field theory with central charge $c[\nu = 0] = 1$, a compactified radius R determined by the dual statistical parameter $R^2 = 1/\tilde{\nu} = e^{2\varphi}$ and $\varphi = \ln R$. The radius R expresses an angular invariance of the theory.

Finally, we would like *to emphasize* that our results were obtained independently of any bosonization idea such as that discussed in [6]. On the contrary, we have used a *symmetry principle*, i.e. the concept of *duality symmetry* between the universal classes h of particles. On the other hand, our definition of *fractal statistics* differs from the notion of *exclusion statistics* [7], in that in our approach we consider *ab initio* the spin-statistics relation $\nu = 2s$ for particles with rational or irrational spin quantum number s. This claim is supported by the *fractal spectrum* which is a *mirror symmetry* in our formulation. Another consequence is the *fractal supersymmetry* which is realized, for example, in the context of the fractional quantum Hall effect [1].

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